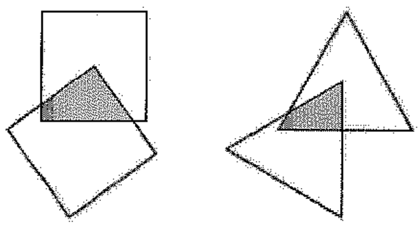
# 2011 John O'Bryan Mathematical Competition
## Junior-Senior 5-person Team Test

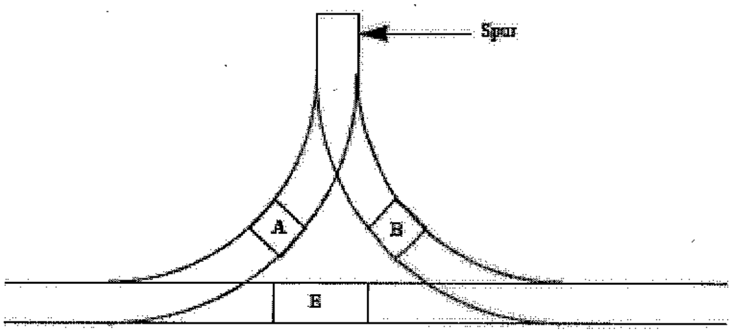
Abbreviated Instructions: Answer each of the following questions **using separate sheet(s) of paper for each numbered problem**. Place your team letter in the upper right corner of each page that will be turned in (failure to do this will result in no score). Problems are equally weighted; **teams must show complete solutions (not just answers) to receive credit**. More specific instructions are read verbally at the beginning of the test.

1. Two identical squares overlap with the corner of one square at the center of the other.

   a. Find the proportion of the area of each square that is in the overlapped section.
   b. Prove that the proportion of area in part (a) is independent of the relative positions of the two squares.
   c. If we replace the squares by identical equilateral triangles, with the corner of one at the center of the other, is the overlap still independent of relative positioning? Explain!



2. Two railroad cars (labeled A and B) are in a siding as shown. An engine (labeled E) is on the main track as shown. Either end of the engine can hitch to either end of each car to pull (or push) it. Of course the cars may be hitched together as well. While each car will fit alone onto the spur, there is not room for two cars, nor for the engine, on the spur. Can the engine switch the position of the two cars while also ending at the same location on the main track where it started?




3. The Fibonacci numbers begin 1, 1, 2, 3, 5, 8, ..., where each element is the sum of the previous two elements.

   a. Which Fibonacci numbers are even?
   b. Prove your result from part (a).
   c. Which Fibonacci numbers have 3 as a factor?
   d. Prove your result from part (c).

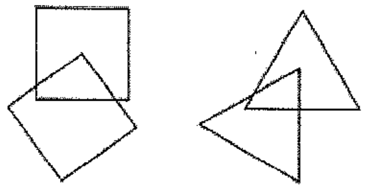
4. As a decimal, the fraction $1/97$ has a repetend (the part that repeats) that begins after the decimal point and is 96 decimals long. If the last three digits of the repetend are A67, compute A.

5. Let $a$ and $b$ be two randomly chosen positive integers (not necessarily distinct) chosen such $a \le 100$ and $b \le 100$. What is the probability that the units digit of $(3^a + 7^b)$ is 6?

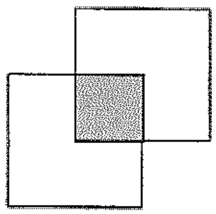
6. What is the largest positive integer $n$ such that 2011! is divisible by $15^n$? Note: 2011! is the product of every positive integer less than or equal to 2011. For example, $6! = 6\times5\times4\times3\times2\times1 = 720$.

7. If $\sin x + \cos x = \frac{1}{2}$, find the value of $(\sin x)^4 + (\cos x)^4$.

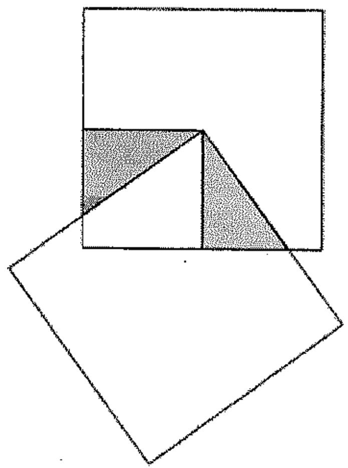
1. Two identical squares overlap with the corner of one square at the center of the other.
   a. Find the proportion of the area of each square that is in the overlapped section.
   b. Prove that the proportion of area in part a is independent of the relative positions of the two squares
   c. If we replace the two word squares by identical equalateral triangles, with the corner of one at the center of the other, is the overlap still independent of relative positioning? Explain

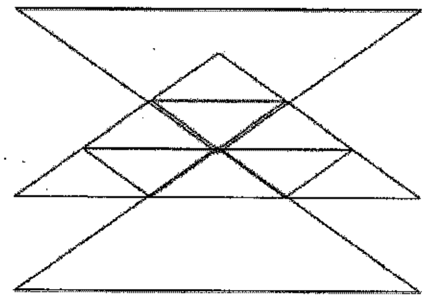


a. By rotating the lower rectangle we see the area is obviously 1/4 of the area of the squares:



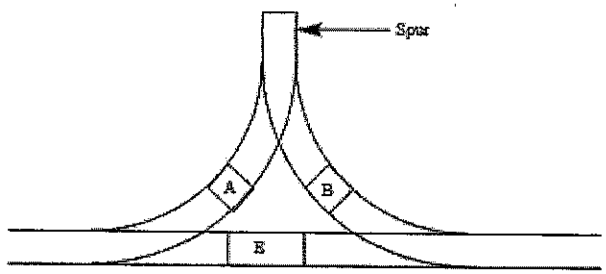
b. In the following picture the two filled triangles have the same area, thus showing that the overlap area is one-quarter of the total area.





c. No, the overlap now changes area. This is easiest seen by looking at triangles inthe two configuarations shown. The top triangle contains twice the overlap area of the bottom triangle. I have trisectected each side of the center triangle and connected all such points with straight lines to form nine congruent triangles.

2. Two railroad cars are in a siding, as shown. Can the engine, E, switch the position of the two cars, with E ending where it started? (While each car will fit alone onto the spur, there is not room for two cars, nor for the engine, on the spur. Either end of the engine can hitch to either end of each car to pull (or push) it. Of course, the cars can be hitched together as well.)




1 Use the engine to move car A down to the straight track and push to the far right.
2. Leaving car A where it is, move the engine up to car B and push it onto the spur.
3. Unhitch car B, move the engine down to the straight track, back to the left and up to where car B waits on the spur.
4. Hitch to car B, pull it down to the straight track and then push it over to car A on the far right. Hitch to car A.
5. Pull both cars back to the left and then push them up towards the spur. Stop with car A on the spur.
6. Unhitch car A (leaving it in the spur) and pull car B down to car A's original position. Unhitch car B.
7. Drive the engine down to the straight track, then far to the right, then up to the spur.
8. Pull car A down to car B's original position.
9. Return the engine to its original position. Relax - you deserve it!

3. The Fibonacci numbers begin 1,1,2,3,5,8,..., where each element is the sum of the two previous elements.

   a. Which Fibonacci numbers are even?

   b. Prove your result from part a.

   c. Which Fibonacci numbers have 3 as a factor?

   d. Prove your result from part c.

a. Just listing out Fibonacci numbers we see an obvious pattern.

{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025}

Every third number (starting with the third) is even.

b. Let $F_n$ denote the $n-$ th Fibonacci number. We need to prove that $F_{3n}$ is even. We proceed by induction on n. We see that $F_3$ is even.

First assuming $F_{3k}$, is even, we must show that $F_{3(k+1)} = F_{3k+3}$ is even. But $F_{3k+3} = F_{3k+2} + F_{3k+1}$. We also have that $F_{3k+2} = F_{3k+1} + F_{3k}$. Subbing this last expression for $F_{3k+2}$ into the previous expression gives $F_{3k+3} = 2F_{3k+1} + F_{3k}$ and since $F_{3k}$ is even, so is $F_{3k+3}$.

Note: An argument using odd+odd=even can also be made.

c. It appears that every fourth Fibonacci number is divisible by 3.

d. An argument like in part b can be made. I include the main computation.

$F_{4(k+1)} = F_{4k+4} = F_{4k+3} + F_{4k+2} = (F_{4k+2} + F_{4k+1}) + (F_{4k+1} + F_{4k}) = F_{4k+2} + 2F_{4k+1} + F_{4k}$. Now replace $F_{4k+2}$ by $F_{4k+1} + F_{4k}$, giving $F_{4k+4} = 3F_{4k+1} + F_{4k}$.

4. As a decimal the fraction $\frac{1}{97}$ has a repetend (the part that repeats) that begins after the decimal point and is 96 decimals long. If the last three digits of the repetend are A67, compute A.

We know that $1/97 = 0.\overline{0** ** ** ***A67}$ - where the stars replace many unknown digits. Since the repetend begins again after the last division, the remainder is a one. Thus 97(0****A67)+1=100...0. Writing out the multiplication we get

$$\begin{array}{r} 0\ldots A\ 6\ 7 \\ \times\ 9\ 7 \\ \hline D\ 6\ 9 \\ 0\ 3\ 0 \\ +1 \\ \hline \ldots 0\ 0\ 0 \end{array}$$

where D = 7A+4. Thus (7A+4)+1 ends in a zero, so 7A ends in a 5. This implies A=5.

5. Let a and b be two - not necessarily distinct - numbers from 1 to 100 (integers). What is the probability that the units digit of $3^a + 7^b$ is 6?

If we look at the units digit of $3^a$, we get the recurring pattern of 3,9,7,1 and for $7^b$ we get 7,9,3,1. ($3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$, etc)

If we sum these possibilities only the pairs (3,3), (9,7) and (7,9) have units digit of 6. So the probability is $\frac{3}{4^2} = \frac{3}{16}$.

6. What is the largest positve integer n such that 2011! is divisible by $15^n$?

Since $15 = 3 \times 5$, we need only find how many times each of these numbers divides into 2011!. Since 5 is the larger of the two factors, we will find out how many 5's there are in 2011! - there will be even more 3's.

5 divides into 2011 402 times.

$5^2$ divides into 2011! 80 times

$5^3$ divides into 2011! 16 times

$5^4$ divides into 2011! 3 times.

So there are 402+80+16+3=501 factors of 5, and so of 15, in 2011!

7. If $\sin x + \cos x = \frac{1}{2}$, find $(\sin x)^4 + (\cos x)^4$.

$\frac{1}{4} = (\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x$. So $\sin x \cos x = \frac{-3}{8}$.

We also have that $1 = (\sin^2 x + \cos^2 x)^2 = \sin^4 x + \cos^4 x + 2\sin^2 x \cos^2 x = \sin^4 x + \cos^4 x + 2(-3/8)^2$.

So, $\sin^4 x + \cos^4 x = \frac{23}{32}$.